Two-particle interference

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The superposition principle leads to coherence phenomena that have no counterpart in classical optics. A *gedanken* experiment, due to Horne and Zeilinger, provides an especially clear illustration of such phenomena, and is presented in a manner suitable to an introductory quantum mechanics course. The experiment displays an interference pattern in the correlation between two particles produced in a momentum-conserving decay, but no interference pattern when either particle is observed separately; it also has interesting Einstein–Podolsky–Rosen-type correlations.

I. INTRODUCTION

Quantum mechanics is the only basic theory of physics that claims to be rigorously linear. This may be read to imply that the superposition principle is the theory's most fundamental postulate. For that reason, simple illustrations of quantum mechanical superposition that are inherently different from the coherence phenomena of classical optics serve an important pedagogical end.[1] An incisive example of such an effect has been demonstrated by Mandel and his collaborators in elegant experiments with two-photon states that display an interference pattern in the correlation between two photons whereas no interference pattern appears when only one photon is observed.[2]

From a pedagogic viewpoint, these experiments are not maximally simple, however. For that reason we discuss a *gedanken* experiment due originally to Horne and Zeilinger,[3] that makes the same point with only a bit of algebra and scalar diffraction theory, and requires no knowledge of spin or polarization. Therefore it is suitable to an introductory course in quantum mechanics. The experiment also has the merit of displaying correlations that have characteristic Einstein–Podolsky–Rosen features.

II. TWO-PARTICLE INTERFEROMETRY

Consider two particles a and b described by a wave function $\Psi(\mathbf{r}_a\mathbf{r}_b;t)$. The probability for detecting a at $\mathbf{r}_a$ and b at $\mathbf{r}_b$ *in coincidence* is

$$P_{ab}(\mathbf{r}_a\mathbf{r}_b;t)=|\Psi(\mathbf{r}_a\mathbf{r}_b;t)|^2. \tag{1}$$

The one-particle probability distributions for detecting a when b is *not* observed at all is

$$P_a(\mathbf{r}_a;t)=\int d^3r_b P_{ab}(\mathbf{r}_a\mathbf{r}_b;t), \tag{2}$$

and similarly for $P_b(\mathbf{r}_b;t)$.

Our purpose is to establish the following.

> In any experimental setup that allows the two particles to traverse different paths, and in which *it is possible, in principle, to determine the path taken by one particle by some observation on the other*, neither particle will, by itself, display an interference pattern (i.e., in P_a or in P_b), but there may be an interference pattern in the $a-b$ coincidence rate P_{ab}, i.e., in the correlation of positions for a and b. On the other hand, if the setup is such that *no observation on one particle can, in principle, determine the path of the other*, then either particle by itself, or both, may display an interference pattern (i.e., in P_a and/or P_b), but there will be no interference pattern in the correlation P_{ab}.

The words ''in principle'' allude, as we will see, to the fact that whether or not an observation that actually determines a path is made does not matter—*what matters is whether such an observation is possible at all.*

The state that will be analyzed to establish these contentions describes the particles produced in a decay process $A \to a+b$, and the experimental setup is the two-particle interferometer shown in Fig. 1.

The interferometer consists of two parallel opaque screens S_a and S_b, each pierced by two pinholes symmetrically placed about the axis normal to the screens, and two parallel detection screens D_a and D_b sensitive only to a and b, respectively. The detectors record the coordinates of particles striking them in coincidence, i.e., determine the joint probability distribution P_{ab}.

This thought experiment is not far-fetched because there are several real-life examples of A. One is positronium, the bound electron–positron system, whose ground state annihilates into two photons; another is the neutral kaon K^0, a particle that decays into two π mesons.

In the process $A\to a+b$ momentum is conserved, and so the decay products (daughters) will go in exactly opposite directions with momenta $\hbar k$ *provided* A was at rest. Then if a passes through one of the two holes on the right, b must pass through the diametrically opposed hole on the left, and therefore a determination of the path of one determines that of the other. However, if A is at rest its position is totally uncertain. Conversely, if A is at the exact center of the setup, its momentum would be totally uncertain, there would be no correlation between the directions of a and b, and so an observation on one daughter would not determine the path of the other.

Hence A's spatial localization s must exceed some lower limit to assure that the daughters can only pass through one pair of diametrically opposed pinholes. This limit is set by the uncertainty principle and momentum conservation. The former states that A's momentum uncertainty satisfies $\Delta p_A \gtrsim \hbar/s$; the latter that the spread in angles Θ between the daughters' momenta is of order $\Delta p_A/\hbar k$, so that $\Theta \gtrsim (1/sk)$. But if the source A is to only illuminate one or the other of the opposed holes, Θ must be much smaller than φ,

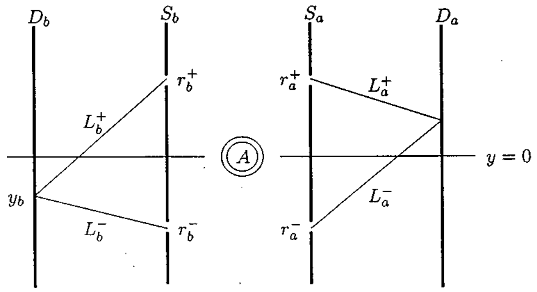

Fig. 1. The parent A is in a state centered on the origin O and having zero mean momentum. It undergoes the momentum-conserving decay $A \to a + b$, with a detected on the right-hand plane D_a, and b on D_b, at the points y_a and y_b, respectively. To reach the detectors, the daughters must pass through one or another pinhole in two parallel screens, S_a and S_b, at equal distance from O. The lengths $L_{a,b}^{\pm}$ enter into the amplitude $\Psi(y_a y_b)$ for coincidence, Eq. (5).

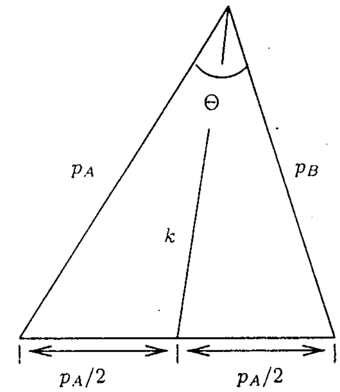

Fig. 3. The departure Θ from back-to-back decay. From classical kinematics, $\langle \Theta^2 \rangle \simeq \langle p_A^2 \rangle / 2(\hbar k)^2$, an estimate which is confirmed by a quantum mechanical calculation.

the angle subtended by the two pinholes on one screen as seen from A (see Figs. 2 and 3). Hence the condition on A's localization is

$$s \gg \frac{1}{k\varphi}. \tag{3}$$

Here we assume that the energy release in the decay is large enough so that both daughters have momenta of approximately the same magnitude $\hbar k$ when (3) is satisfied.

We now turn to the daughters' wave function $\Psi_{\text{out}}(\mathbf{r}_a \mathbf{r}_b)$ outside the screens S_a and S_b. As the particles do not interact in this region, Ψ_{out} must be a linear combination of products of one-particle wave functions, with each such function a spherical wave $\psi(r) = e^{ikr}/r$ emanating from one of the pinholes. When Ψ_{out} is evaluated at the detection screens, one such term is $\psi(L_a^+)\psi(L_b^-)$ for the case where a emerges from the upper hole on the right and b from the lower one on the left, with the various distances $L_{a,b}^{\pm}$ from the pinholes to the detectors defined in Fig. 1. In general, Ψ_{out} is a linear combination of four such products, with four arbitrary coefficients determined by matching Ψ_{out} at the pinholes to the wave function $\Psi_{\text{in}}(\mathbf{r}_a \mathbf{r}_b)$ in the interior region between the screens S_a and S_b.

The state of A is assumed to be a spherically symmetric wave packet of size s and zero mean momentum centered at the origin. This state is symmetric under reflection through the $y=0$ plane shown in Fig. 1 and, provided that the interaction responsible for the process $A \to a + b$ is invariant under rotations, $\Psi_{\text{in}}(\mathbf{r}_a \mathbf{r}_b)$ also has this reflection symmetry. Hence the values of Ψ_{in} at the four pinholes are given by just two complex numbers:

$$\begin{aligned} \Psi_{\text{in}}(\mathbf{r}_a^+ \mathbf{r}_b^-) &= \Psi_{\text{in}}(\mathbf{r}_a^- \mathbf{r}_b^+) = \alpha, \\ \Psi_{\text{in}}(\mathbf{r}_a^+ \mathbf{r}_b^+) &= \Psi_{\text{in}}(\mathbf{r}_a^- \mathbf{r}_b^-) = \beta, \end{aligned} \tag{4}$$

where $\mathbf{r}_{a,b}^{\pm}$ are the positions of the pinholes.

These coefficients have a simple meaning: $|\alpha|^2$ is the probability for a and b to pass through diametrically opposed holes, i.e., for A to have undergone back-to-back decay, whereas $|\beta|^2$ is the probability for both to pass through either the two upper or the two lower holes. Clearly, $|\beta/\alpha|^2 \ll 1$ if the initial state of A satisfies the source size condition, Eq. (3).

The outside wave function evaluated at the detectors, when (4) holds, is

$$\begin{aligned} \Psi_{\text{out}} \doteq\, & \alpha(e^{ikL_a^+} e^{ikL_b^-} + e^{ikL_a^-} e^{ikL_b^+}) \\ & + \beta(e^{ikL_a^+} e^{ikL_b^+} + e^{ikL_a^-} e^{ikL_b^-}). \end{aligned} \tag{5}$$

Here the distance L_0 from the screens to the detectors is assumed to be much larger than that between the holes on either screen, so that the denominators in e^{ikr}/r can all be replaced by L_0, and have then been absorbed into an irrelevant overall factor; in (5) and henceforth, $\doteq$ means equal apart from such a factor. In this geometry (the Fraunhofer diffraction limit), the various lengths can be approximated by

$$L_a^{\pm} = L_0 \mp \theta y_a, \quad L_b^{\pm} = L_0 \mp \theta y_b, \tag{6}$$

where the y coordinates and θ are defined in Figs. 1 and 2. With these small-angle approximations, (5) simplifies to

$$\Psi_{\text{out}}(y_a y_b) \doteq \alpha \cos[k\theta(y_a - y_b)] + \beta \cos[k\theta(y_a + y_b)]. \tag{7}$$

For general values of α/β, this is *an entangled state*, one that cannot be expressed as a single product of one-particle wave functions, $\psi_a(y_a)\psi_b(y_b)$.

The two-particle interference phenomena require, as we have already intimated, that the decay be back-to-back, that is, $|\beta/\alpha|^2 \simeq 0$. The joint probability distribution for detecting a at y_a and b at y_b in coincidence is then

$$P_{ab}(y_a y_b) \doteq |\cos[k\theta(y_a - y_b)]|^2. \tag{8}$$

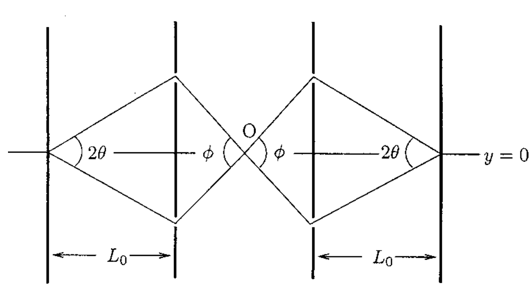

Fig. 2. The distances and angles that define the arrangement in Fig. 1.

This states that *the coincidence rate will display an interference pattern in the variable* $|y_a-y_b|$, *the "distance," so to say, between locations on the widely separated detection screens*.

In striking contrast, *the distributions of locations of particles on the individual detection screens show no interference pattern*. This is so because the probability for detecting a at y_a, regardless of where b struck the other detector, is

$$P_a(y_a)=\frac{1}{2Y}\int_{-Y}^{Y}dy_b\,P(y_ay_b)=\text{const}+O(1/Y). \qquad (9)$$

This distribution is independent of a's position, a result that requires the size $2Y$ of the detection region for b to be large enough to yield no information about b's position, i.e., $Y \gg (1/k\theta)$, the distance between interference fringes.

The existence of an interference effect in the coincidence rate when there is none in the individual rates is a quantum mechanical phenomenon. Hence it is important to understand why this initially surprising result is, in retrospect, not surprising!

The absence of an interference pattern in the individual rates is due to the possibility of determining the path that both particles took by an observation on just one. To accomplish this one can, for example, replace b's position detector D_b by a device that determines b's momentum along y as it arrives at the left-hand detection plane. This determines which hole in S_b it traversed and, because of the back-to-back decay, which hole in the other screen was traversed by a. Hence there can be no interference pattern in the locations of a's position, for that would require a coherent addition of amplitudes from the two holes on the right side, which is excluded by the knowledge of which hole a traversed. Of course, the same conclusion holds for b.

The argument of the preceding paragraph might lead one to suspect that a gross error has been committed in exploiting results from two distinct experiments measuring incompatible observables. That is not so, however: It is the Einstein–Podolsky–Rosen (EPR) feature of this experiment, as will be made clear in Sec. IV.

To shed further light on what has just been discussed, consider what happens if the back-to-back decay does not dominate, and β is not negligible. In particular, consider the case where the source size s is small enough so that the two holes on either side are illuminated equally, i.e., $\alpha=\beta$. Then (7) becomes

$$\Psi_{\text{out}}(y_ay_b)\doteq\cos(k\theta y_a)\cos(k\theta y_b). \qquad (10)$$

This is not an entangled state. It describes *independent* diffraction patterns on each of the two detection screens. There are no correlations in either coordinates or momenta because determining which hole is traversed by one particle does not determine the path the other took.

The interference pattern in the coincidence rate when back-to-back decay dominates (i.e., $|\alpha|\gg|\beta|$) is due to the correlations in the momenta of a and b. At the detecting planes the particles have y components of momentum, p_a and p_b, that are either $\hbar k\theta$ or $-\hbar k\theta$. At each detector there is a random distribution of events with $\pm\hbar k\theta$, but a *strict correlation* between those on the right and left: If one particle has momentum $\hbar k\theta$, the other is *guaranteed* to have $-\hbar k\theta$. The coordinate space wave function

$$\Psi_{\text{out}}(y_ay_b)\doteq\cos[k\theta(y_a-y_b)]$$

is the Fourier transform of a momentum space wave function $\Phi(p_ap_b)$ which expresses the preceding sentence in precise terms, namely

$$\Phi\doteq\delta(p_a-\hbar k\theta)\,\delta(p_b+\hbar k\theta)+\delta(p_a+\hbar k\theta)\,\delta(p_b-\hbar k\theta). \qquad (11)$$

III. THE TRANSITION FROM ONE- TO TWO-PARTICLE INTERFERENCE

The two cases already discussed in detail, one where there are independent one-particle diffraction patterns and no interference effects in the coincidence rate, and the other where there is an interference effect in the coincidence rate but no one-particle diffraction patterns, are limiting cases of the general situation described by Eq. (5) for arbitrary values of $|\beta/\alpha|\equiv\gamma$. The quality that changes along the continuum from $\gamma=0$ to $\gamma=1$ is the degree of confidence with which it is possible to determine the path of one particle by an observation on the other. If $\gamma=0$, it is known for sure which path a took from a momentum measurement on b, but as γ increases this becomes progressively less certain, and the two possible paths become equally probable when $\gamma=1$. The transition from two-particle to one-particle interference goes hand-in-hand with this decrease of knowledge attainable *in principle*.

It is instructive to phrase the preceding paragraph in more general terms. Let $\phi_n(a;q_a)$ be a wave function of system a, whose coordinates are collectively designated by q_a, and similarly for b. In the example of Sec. II, at the detectors

$$\phi_\pm(a;q_a)=e^{ikL_a^\pm}, \quad \phi_\pm(b;q_b)=e^{ikL_b^\pm}. \qquad (12)$$

Consider now the generic entangled state

$$\Psi(q_aq_b)=\phi_n(a;q_a)\,\phi_m(b;q_b)+\phi_{n'}(a;q_a)\,\phi_{m'}(b;q_b). \qquad (13)$$

In the probability distribution, this produces the two-particle interference term

$$I_{ab}(q_a,q_b) =2\,\text{Re}\{\phi_n(a;q_a)\,\phi^*_{n'}(a;q_a)\,\phi_m(b;q_b)\,\phi^*_{m'}(b;q_b)\}. \qquad (14)$$

If b is not detected, the observable quantity is

$$I(q_a)=2\,\text{Re}\{V\phi_n(a;q_a)\,\phi^*_{n'}(a;q_a)\}, \qquad (15)$$

where V, which we may call the visibility amplitude, is

$$V=\int dq_b\phi_m(b;q_b)\,\phi^*_{m'}(b;q_b). \qquad (16)$$

Thus a one-particle interference pattern will only be visible if the states ϕ_m *and* $\phi_{m'}$ *of particle b are not orthogonal*.

In the example of Sec. II, the two states of b that interfere at the detector D_b are orthogonal: They are states of differing momentum along the y direction. Hence there is no interference in the one-particle probability distribution.

A somewhat different *gedanken* experiment elucidates the point made in the discussion leading to Eq. (16). Assume b is charged, as in $K_S\rightarrow\pi^++\pi^-$, and that the screens on the left-hand side of Fig. 1 are replaced by magnetic traps that leave b in one or the other of two states $\phi_{\pm}(b)$ if a has the momentum required to pass through the corresponding right-hand hole $\mathbf{r}_a^{\mp}$. Then there will be an interference pattern in the probability distribution of y_a as long as there is spatial overlap between these trapped states of b, because then a measurement on b does not provide an *unambiguous* determination of the path of a.

This modified experiment serves to emphasize what is already implicit in (15) and (16): *The visibility of the interference pattern displayed by a alone when it is in an entangled two-body state is determined by the confidence with which an observation on b can determine the state of a*. It need not be an either-or situation, as it is in the interferometer of Fig. 1 when $\beta=0$.

IV. THE EPR FEATURE

Our two-particle interferometer can display the ''spooky action-at-a-distance'' feature first discussed by Einstein, Podolsky, and Rosen (EPR). There is, of course, nothing novel about using a particular measurement on one particle to determine a property of another that is, at the time, arbitrarily far away. If it is known that a missile with a known momentum will separate into two pieces, it suffices to determine the momentum of one piece to determine that of the other. But here there are features that are totally foreign to classical physics. The classical momentum correlations in a missile breakup would be described by Eq. (11) if it were taken to be *the joint probability distribution*, whereas in quantum mechanics this expression is *the probability amplitude*. The wave-like correlation in positions between particles at arbitrarily large separations is a consequence of this coherent superposition in momentum space, a concept that does not exist in classical mechanics.

In the original EPR example, and in the more familiar and practical version due to Bohm in which spins or photon polarizations are observed, there are strict correlations between observables at space-like separations. In this example, there are only statistical EPR-type correlations, but they too can convey the misconceptions that instantaneous signaling is possible or that hidden variables are at work.

The EPR feature in our experiment is that a *free choice* between determining the position or momentum of b determines the diffraction pattern displayed by the distant particle a. This feature is not visible with arbitrarily small pinholes, because such an idealized aperture produces no angular variation in intensity. With pinholes having a finite aperture $2d$, the one-particle distribution is no longer uniform, as it is in Eq. (9), but becomes

$$P_a(y_a) \doteq \left|\frac{\sin k\epsilon(\rho-y_a)}{k\epsilon(\rho-y_a)}\right|^2+\left|\frac{\sin k\epsilon(\rho+y_a)}{k\epsilon(\rho+y_a)}\right|^2, \tag{17}$$

where $\epsilon=d/L_0$, $\rho=l+\frac{1}{2}\varphi L_0$, and $2l$ is the distance between the pinholes. This is the incoherent sum of two conventional one-hole diffraction patterns with centers separated by 2ρ. These diffraction patterns will overlap strongly if ϵ is sufficiently small, as will be assumed.

The EPR correlations can be demonstrated by the following experiment. On the right, the apparatus always measures the coordinate y_a of a, while on the left the apparatus switches at random between measurements of position or momentum of b. All measurements are done in coincidence. This does not require communication between the widely separated ''laboratories'' D_a and D_b; there can be a protocol to insert only one specimen of A at regularly spaced intervals sufficiently long to insure that both laboratories carry out observations on the same $a+b$ specimen. After the run is over, the list of observations that were made on b is transmitted to the other distant laboratory, where the data on the coordinate y_a are separated into three sets: (1) those where $p_b=\hbar k\theta$, (2) those where $p_b=-\hbar k\theta$, and (3) those where y_b was measured. No human intervention is necessary after the apparatus is set to work; the collection, transmission, and processing of the data can be fully automated.

The prediction of quantum mechanics is that set (1) will display the diffraction pattern in y_a from the upper hole alone [the first term of Eq. (17)]; set (2), the diffraction pattern from the lower hole (the second term); and set (3), the oscillating correlation function $|\cos[k\theta(y_a-y_b)]|^2$. While this particular experiment has not been done, enough experiments of the EPR variety have been so that there is no reason to doubt that what has just been claimed would be confirmed.

As is well known, such a setup provides no means for instantaneous signaling between the two laboratories, though in this particular example the ''spookiness'' is less dramatic than in the EPR–Bohm examples because here the correlations between observations at space-like separations are only statistical. Nevertheless, before the information concerning the sequence of observations on b reaches the laboratory that determined the coordinates y_a, the latter only sees a quite featureless distribution of hits that gives no hint of the correlation and diffraction patterns that can be extracted once the information about b is in hand.

The interference patterns displayed, or not displayed, by the two-particle wave function are, to underscore it yet again, determined by what the experimenter can in principle do, and not by what is actually done. This is best brought out by the *delayed choice* property—by the freedom to decide whether the momentum or coordinate is to be determined just before the particles are actually detected. What counts is whether the option to make this choice exists, and not whether the option is exercised. If the wave function is to pass the delayed choice test, it must possess a sufficient richness of properties to cope with all measurements that could be done in the future. It must, in this example, be able to display itself in a wave-like or particle-like guise, or in various combinations of these guises, depending, so to say, on what question will be asked of it.

ACKNOWLEDGMENTS

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SCIENTIFIC OPERA

The more we get into the niceties of the scientific literature, the more extraordinary it becomes. It is now a real opera. Crowds of people are mobilised by the references; from offstage hundreds of accessories are brought in. Imaginary readers are conjured up which are not asked only to believe the author but to spell out what sort of tortures, ordeals and trials the heroes should undergo before being recognised as such. Then the text unfolds the dramatic story of these trials. Indeed, the heroes triumph over all the powers of darkness, like the Prince in *The Magic Flute*. The author adds more and more impossible trials just, it seems, for the pleasure of watching the hero overcoming them. The authors challenge the audience and their heroes sending a new bad guy, a storm, a devil, a curse, a dragon, and the heroes fight them. At the end, the readers, ashamed of their former doubts, have to accept the author's claim. These operas unfold thousands of times in the pages of *Nature* or the *Physical Review* (for the benefit, I admit, of very, very few spectators indeed).

Bruno Latour, *Science in Action—How to Follow Scientists and Engineers Through Society* (Harvard University Press, Cambridge, Massachusetts, 1987), pp. 53–54.